Weekend Activity: Power Series! Solutions

1) $1+x+x^2+x^3+x^4+x^5+x^6$

2) $\displaystyle\sum_{n=0}^{\infty} 2^n$ diverges, since it's a geometric series with $r=2$

3) $\displaystyle\sum_{n=0}^{\infty}(\frac{-1}{2})^n$ converges, since it's a geometric series with $r=\frac{-1}{2}$

4) Plugging in any value $x=r$ will yield a geometric series with common ratio r. Thus the series converges if and only if $-1<x<1$

5) $f(x)=\frac{1}{1-x}$

6) The sum when $x=3$ is very large compared with what we would expect given $f(3)$. This is because 3 is not in the interval of convergence.

6) $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}$

7) If $x=0$, the series converges since it is a sum of zeroes. If $x\neq 0$, we look at the limit of $\frac{a_{n+1}}{a_n}$: $\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}=\lim_{n\to\infty}\frac{x}{n+1}=0$. Thus the series converges for all x by the ratio test.

8) a) $0+1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$

b) The derivative of $s_n=\displaystyle\sum_{i=0}^{n}\frac{x^i}{i!}$ is $s_{n-1}=\displaystyle\sum_{i=0}^{n-1}\frac{x^i}{i!}$ c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

9) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

10) The answers to 8 and 9 were both equal to the original power series. $f(x)=e^x$ since this is the only function equal to its integral and derivative.

11)$y=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2$

12) $y=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{6}f'''(0)x^3$

13) $y=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{6}f^{(3)}(0)x^3+\frac{1}{24}f^{(4)}(0)x^4$

14) $P_n=\sum_{i=0}^{n}\frac{f^{(i)}(0)\,x^i}{i!}$

15) $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)\ x^n}{n!}$ is the general limit of P_n. Since $f(x)=e^x$, $f^{(n)}(0)=1$ for all n, thus the series is $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. This series converges everywhere to e^x, as was shown previously.

16*) $\lim_{n\to\infty}P_n=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}=f(x)$ when the series converges.